Territory covered by N Lévy flights on d-dimensional lattices

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We study the territory covered by N Lévy flights by calculating the mean number of distinct sites, $\langle S_N(n)\rangle$, visited after n time steps on a d-dimensional, $d\geqslant 2$, lattice. The Lévy flights are initially at the origin and each has a probability $A\ell^{-(d+\alpha)}$ to perform an ℓ-length jump in a randomly chosen direction at each time step. We obtain asymptotic results for different values of α. For $d=2$ and $N\to\infty$ we find $\langle S_N(n)\rangle \propto C_\alpha N^{2/(2+\alpha)}n^{4/(2+\alpha)}$, when $\alpha<2$ and $\langle S_N(n)\rangle \propto N^{2/(2+\alpha)}n^{2/\alpha}$, when $\alpha>2$. For $d=2$ and $n\to\infty$ we find $\langle S_N(n)\rangle \propto Nn$ for $\alpha<2$ and $\langle S_N(n)\rangle \propto Nn/\ln n$ for $\alpha>2$. The last limit corresponds to the result obtained by Larralde *et al.* [Phys. Rev. A **45**, 7128 (1992)] for bounded jumps. We also present asymptotic results for $\langle S_N(n)\rangle$ on $d\geqslant 3$ dimensional lattices. [S1063-651X(97)04102-0]

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I. INTRODUCTION

Lévy flights are a model for diffusion, where the displacement of a single particle has a distribution, called Lévy distribution, with infinite variance, and sometimes even infinite means. Lévy flights have been found useful in describing the enhanced dynamics of nonlinear, chaotic, turbulent, and biological systems, in general, systems with enhanced diffusion compared to Brownian motion [1–8]. For more recent results on Lévy flights see [9].

Lévy distributions belong to the domain of attraction of the stable laws with an infinite variance [10,11]. The probability of making an ℓ-length jump is asymptotically

$$p(\ell)\propto \ell^{-(d+\alpha)}, \quad \alpha<2 \tag{1}$$

in the limit $\ell\to\infty$. This distribution possesses the following characteristic property. If we assume that L is the sum of Lévy random variables, $L=\Sigma_{i=1}^{n}\ell_i$, then it has a distribution similar to Eq. (1),

$$p_n(L)\propto \frac{n}{L^{d+\alpha}}. \tag{2}$$

It is easy to see that mean square $\langle L^2\rangle$ is infinite for all n. One can also note that power-law behavior of the tails of the distribution function, Eq. (2), implies the relatively high probability large values of displacement L, explaining the name ''flights.''

The territory covered by a diffusing particle is an important characteristic of its dynamics. This quantity is found to be useful in ecology, chemical reactions, and spreading phenomena [12–17]. In the case of a random walk on the lattice this amount corresponds to the number of distinct sites visited by a single random walker $S_1(n)$. The properties of this quantity have been studied in a large number of works for bounded step random walks, Lévy flights and Lévy walks [18–38]. Its natural generalization, the average number of distinct sites visited by N random walkers, $\langle S_N(n)\rangle$, for bounded steps random walks has been studied by Larralde *et al.* in [39]. In a recent work [40], a similar approach has been developed and applied to N Lévy flights restricted to the one-dimensional lattice.

In this paper we study the *average* number of distinct sites $\langle S_N(n)\rangle$, visited by N Lévy flights each having a probability $A\ell^{-(d+\alpha)}$ to perform an ℓ-length jump on d-dimensional lattice. The direction of the jump is chosen randomly. Here we assume that Eq. (1) is the base distribution law but extend it for all values of α. Typical realizations of the distinct sites visited for several values of α are shown in Fig. 1.

We find that although for $\alpha>2$ the second moments are finite and the distribution is no longer in the domain of attraction of the infinite variance stable laws, $\langle S_N(n)\rangle$ of Lévy flights behave qualitatively differently from that of the bounded step walk.

II. ANALYTICAL APPROACH

For simplicity the analysis is presented in detail for $d=2$, generalizations for $d\geqslant 3$ are discussed in Sec. VII.

For different values of α we can approximate the probability of a Lévy flight to be at lattice site $\mathbf{r}$ at step n by two different limit distribution laws. For $|\mathbf{r}|\gg n^{1/2},\alpha>2$ and for $|\mathbf{r}|\gg n^{1/\alpha},\alpha\leqslant 2$ the appropriate approximation for the probability to be at a lattice site $\mathbf{r}$ at step n is readily shown to be the stable law

$$p_n(\mathbf{r})=n^{-2/\alpha}p(n^{-1/\alpha}\mathbf{r})\propto n|\mathbf{r}|^{-(2+\alpha)}. \tag{3}$$

For $\alpha>2$ and $|\mathbf{r}|\ll n^{1/2}$ the function $p_n(\mathbf{r})$ behaves as the Gaussian,

$$p_n(\mathbf{r})\propto n^{-1}\exp(-|\mathbf{r}|^2/\sigma^2 n). \tag{4}$$

We consider the function $\langle S_N(n)\rangle$ as the sum $\langle S_N(n)\rangle=\langle S_N(n)\rangle^-+\langle S_N(n)\rangle^+$. The first term in the sum corresponds to the distinct sites visited inside the circle $|\mathbf{r}|\leqslant n^{1/\gamma}$ and the second term corresponds to the sites visited in the exterior of this circle: i.e., in the area $|\mathbf{r}|>n^{1/\gamma}$, where

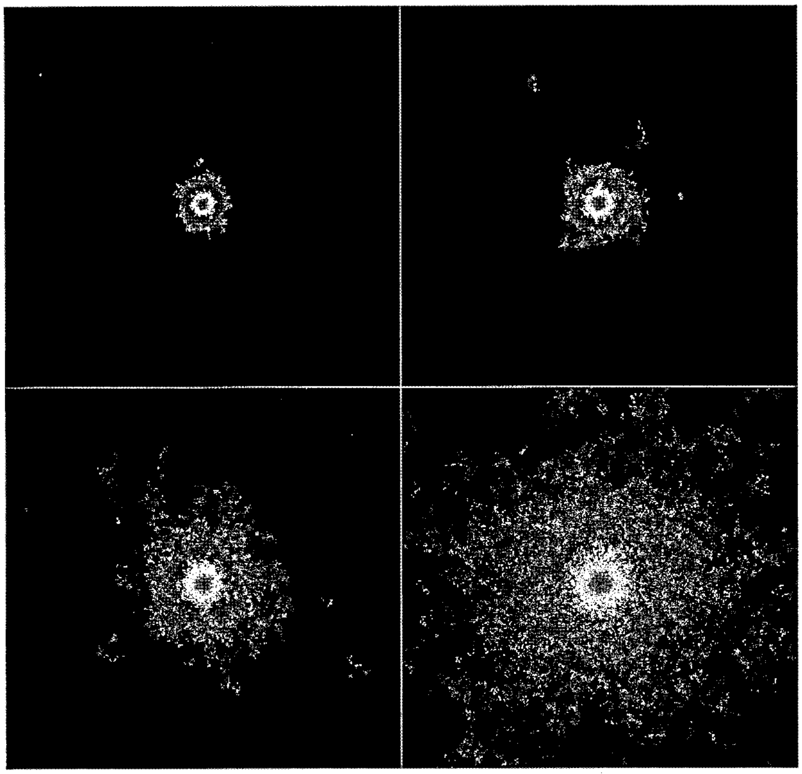

FIG. 1. Snapshots of the territory covered by $N=1000$ Lévy flights for different values of α ($\alpha=1.5,2.0,2.5,3.5$) after the same time. The different gray colors correspond to the sequence of times $n=2,25,100,225,400,500$.

$$\gamma=\begin{cases}\alpha, & \alpha\leq 2\\ 2, & \alpha>2.\end{cases}$$

Thus for each term we can use the corresponding $p_n(\mathbf{r})$ approximation, Eqs. (3) and (4).

We begin by introducing the notation, which is similar to that developed by Larralde *et al.* [39] for bounded jumps and used by Berkolaiko *et al.* [40] for Lévy flights in one dimension. Let us denote by $f_n(\mathbf{r})$ the probability that a site $\mathbf{r}$ will be first visited at step n by a single Lévy flight initially at the origin. We denote by $\Gamma_n(\mathbf{r})$ the probability that site $\mathbf{r}$ has not been visited by any Lévy flight by step n. This function is related to the $f_n(\mathbf{r})$ by

$$\Gamma_n(\mathbf{r})=1-\sum_{k=1}^{n} f_k(\mathbf{r}). \tag{5}$$

The probability that a site $\mathbf{r}$ has been visited by at least one of the N Lévy flights in the course of n steps is $1-\Gamma_n^N(\mathbf{r})$. Thus the expected number of distinct sites visited by the N Lévy flights by the nth step is $\langle S_N(n)\rangle$,

$$\langle S_N(n)\rangle=\sum_{\mathbf{r}}\,[1-\Gamma_n^N(\mathbf{r})], \tag{6}$$

where the sum is over all sites of the lattice. According to the above definition the functions $\langle S_N(n)\rangle^-$ and $\langle S_N(n)\rangle^+$ can be presented as

$$\langle S_N(n)\rangle^-=\sum_{|\mathbf{r}|\leq n^{1/\gamma}}[1-\Gamma_n^N(\mathbf{r})], \tag{7}$$

$$\langle S_N(n)\rangle^+=\sum_{|\mathbf{r}|>n^{1/\gamma}}[1-\Gamma_n^N(\mathbf{r})]. \tag{8}$$

The technique presented below allows one to get the approximation for the number of sites visited in the area $|\mathbf{r}|>n^{1/\gamma}$, i.e., $\langle S_N(n)\rangle^+$. In the limit $N\to\infty$ function $\langle S_N(n)\rangle^-$ is bounded by the number of sites inside the circle $|\mathbf{r}|\leq n^{1/\gamma}$. Thus $\langle S_N(n)\rangle^-$ is bounded for fixed n and the leading term contributing to $\langle S_N(n)\rangle$ is $\langle S_N(n)\rangle^+$, i.e.,

$$\langle S_N(n)\rangle\approx\langle S_N(n)\rangle^+,\quad N\to\infty. \tag{9}$$

Other arguments will be employed in the limit $n\to\infty$ to get the final results for $\langle S_N(n)\rangle$. This limit will be discussed in Sec. VI.

To study the behavior of the function $f_n(\mathbf{r})$ we introduce the following generating functions with respect to the step number,

$$p(\mathbf{r},z)\equiv\sum_{n=0}^{\infty} p_n(\mathbf{r})z^n,\quad f(\mathbf{r},z)\equiv\sum_{n=0}^{\infty} f_n(\mathbf{r})z^n.$$

One can note the relation [38] between $f(\mathbf{r},z)$ and $p(\mathbf{r},z)$,

$$f(\mathbf{r},z)=\frac{p(\mathbf{r},z)}{p(0,z)},\quad \mathbf{r}\neq 0. \tag{10}$$

In the following we present the formalism to calculate $p(\mathbf{r},z)$ and $p(0,z)$ in order to get the singular behavior of $f(\mathbf{r},z)$ and therefore the form of $f_n(\mathbf{r})$.

To calculate the asymptotic form of $\langle S_N(n)\rangle^+$ we consider $p_n(\mathbf{r})$ for the $|\mathbf{r}|>n^{1/\gamma}$ regime, at which

$$p_n(\mathbf{r})\propto n|\mathbf{r}|^{-(2+\alpha)}.$$

Thus calculation of the generating function $p(\mathbf{r},z)$ near $z=1$ yields

$$p(\mathbf{r},z)\propto \frac{1}{(1-z)^2|\mathbf{r}|^{2+\alpha}}.$$

For $p(0,z)$ one has an asymptotic formula [37,38] for $z\to 1$,

$$p(0,z)\propto \frac{1}{\pi^2}\int\int_0^{\pi}\frac{d\theta_1 d\theta_2}{1-az\hat{p}(\boldsymbol{\theta})}. \tag{11}$$

Substitution of the Fourier transform $\hat{p}(\boldsymbol{\theta})$ of the function $p(\mathbf{r})$,

$$\hat{p}(\boldsymbol{\theta})\approx\begin{cases}1-C|\boldsymbol{\theta}|^{\alpha}, & \alpha\neq 2\\ 1-C|\boldsymbol{\theta}|^2\ln(|\boldsymbol{\theta}|)-C_1|\boldsymbol{\theta}|^2, & \alpha=2\end{cases}$$

into Eq. (11) yields

$$p(0,z)\propto \frac{1}{\pi^2}\int\int_0^{\pi}\frac{d\theta_1 d\theta_2}{1-z+aC|\boldsymbol{\theta}|^{\alpha}[\ln(1/|\theta|)]^{\delta_{\alpha 2}}}, \tag{12}$$

where $\delta_{\alpha 2}$ is the Kronecker delta. We can see that the behavior of $p(0,z)$ is singular for $\alpha\geqslant 2$ and regular for $\alpha<2$.

We consider the case $\alpha>2$ in the next section.

III. THE CASE $\alpha>2$

By introducing into Eq. (12) the substitution $\theta_i=(1-z)^{1/\alpha}\phi_i$ we obtain

$$p(0,z)\propto\frac{(1-z)^{2/\alpha}}{\pi^2(1-z)}\int\int_0^{\pi/(1-z)^{2/\alpha}}\frac{d\phi_1 d\phi_2}{1+a|\boldsymbol{\phi}|^{\alpha}}$$

$$\approx\frac{(1-z)^{2/\alpha}}{\pi^2(1-z)}\int\int_0^{\infty}\frac{d\phi_1 d\phi_2}{1+a|\boldsymbol{\phi}|^{\alpha}},$$

where the upper limit of integration was extended to ∞, since we are interested in the behavior of $p(0,z)$ near $z=1$. Substitution of $p(\mathbf{r},z)$ and $p(0,z)$ into Eq. (10) yields

$$f(\mathbf{r},z)\propto C_{\alpha}\frac{1}{|\mathbf{r}|^{2+\alpha}(1-z)^{1+2/\alpha}},$$

where C_{α} is the constant. By inverting this generating function one obtains

$$f_n(\mathbf{r})\propto C_{\alpha}\frac{n^{2/\alpha}}{|\mathbf{r}|^{2+\alpha}}.$$

A simple calculation gives

$$\Gamma_n(\mathbf{r})=1-\sum_{k=0}^{n}f_k(\mathbf{r})\sim 1-\int_0^n C_{\alpha}\frac{\tau^{2/\alpha}}{|\mathbf{r}|^{2+\alpha}}d\tau$$

$$=1-C_{\alpha}\frac{n^{1+2/\alpha}}{|\mathbf{r}|^{2+\alpha}}$$

$$\approx\exp\left(-C_{\alpha}\frac{n^{1+2/\alpha}}{|\mathbf{r}|^{2+\alpha}}\right),$$

where this approximation is valid, since $|\mathbf{r}|>n^{1/2}\geqslant n^{1/\alpha}$. Therefore, approximating the sum in Eq. (8) by an integral, one has

$$\langle S_N(n)\rangle^+\propto\int\int_{|\mathbf{r}|>n^{1/2}}\left[1-\exp\left(-C_{\alpha}\frac{Nn^{1+2/\alpha}}{|\mathbf{r}|^{2+\alpha}}\right)\right]d\mathbf{r}=n^{2/\alpha}(NC_{\alpha})^{2/(2+\alpha)}\int\int_{|\mathbf{s}|>\rho(n,N)}[1-\exp(|\mathbf{s}|^{-2-\alpha})]d\mathbf{s}, \tag{13}$$

where

$$\mathbf{r}=n^{1/\alpha}(NC_{\alpha})^{1/(2+\alpha)}\mathbf{s},$$

$$\rho(n,N)=n^{(\alpha-2)/2\alpha}N^{-1/(2+\alpha)}.$$

For $N\to\infty$ the lower limit of the integration can be set to 0, yielding

$$\langle S_N(n)\rangle^+\propto n^{2/\alpha}N^{2/(2+\alpha)},\quad N\to\infty. \tag{14}$$

For N fixed and $n\to\infty$ the lower integration limit $\rho(n,N)\to\infty$. Thus substitution of $\rho(n,N)=0$ in the integral (13) gives an upper bound for $\langle S_N(n)\rangle^+$, i.e.,

$$\langle S_N(n)\rangle^+<n^{2/\alpha}N^{2/(2+\alpha)},\quad N\ \text{fixed}. \tag{15}$$

The final asymptotic results for $\langle S_N(n)\rangle$ will be formulated in Sec. VI.

IV. THE CASE $\alpha<2$

In the case $\alpha<2$ the integral in Eq. (12) converges for $z\to 1$, therefore in this limit

$$f(\mathbf{r},z)\propto\frac{p(\mathbf{r},z)}{p(0,1)},$$

thus inverting the generating function $f(\mathbf{r},z)$ one gets

$$f_n(\mathbf{r})\propto p_n(\mathbf{r}).$$

This equation reflects the fact that in this range of α the jumps are so large that a single Lévy flight mostly visits new sites. Using Eq. (5) $\Gamma_n(\mathbf{r})$ has the approximation

$$\Gamma_n(\mathbf{r})\approx\exp\left(-C\frac{n^2}{|\mathbf{r}|^{2+\alpha}}\right)$$

and, substituting $\mathbf{r}=n^{2/(2+\alpha)}N^{1/(2+\alpha)}\mathbf{s}$,

$$\langle S_N(n)\rangle^+ \propto n^{4/(2+\alpha)}(NC_\alpha)^{2/(2+\alpha)} \times \int\int_{|\mathbf{s}|>\rho(n,N)} [1-\exp(|\mathbf{s}|^{-2-\alpha})]d\mathbf{s},$$

where $\rho(n,N)=N^{-1/(2+\alpha)}n^{(2-\alpha)/\alpha(2+\alpha)}$. In the limit $N\to\infty$ the lower limit of integration can be extended to 0, resulting in

$$\langle S_N(n)\rangle^+ \propto n^{4/(2+\alpha)}(NC_\alpha)^{2/(2+\alpha)}, \quad N\to\infty. \tag{16}$$

In the limit $n\to\infty$, the lower limit tends to infinity, therefore we can expand the exponent as $\exp(|\mathbf{s}|^{-2-\alpha})\approx 1-|\mathbf{s}|^{-2-\alpha}$ and integrate as

$$\langle S_N(n)\rangle^+ \propto n^{4/(2+\alpha)}(NC_\alpha)^{2/(2+\alpha)} 2\pi \int_{\rho(n,N)}^{\infty} s^{-1-\alpha}ds \sim Nn, \tag{17}$$

where, as before, $\rho(n,N)=N^{-1/(2+\alpha)}n^{(2-\alpha)/\alpha(2+\alpha)}$.

V. THE CASE $\alpha=2$

In the case of $\alpha=2$ one can approximate $p(0,z)$ as

$$p(0,z)\propto \ln\left(\frac{1}{1-z}\right),$$

which leads to

$$\Gamma_n(\mathbf{r})\approx \exp\left(-C\frac{n^2}{|\mathbf{r}|^4\ln(n)}\right)$$

and

$$\langle S_N(n)\rangle^+ \propto \frac{N^{1/2}n}{[\ln(n)]^{1/2}}, \quad N\to\infty. \tag{18}$$

For the limit $n\to\infty$ one can find the expression for the $\langle S_N(n)\rangle^+$ in complete analogy to derivation of Eq. (17). This yields

$$\langle S_N(n)\rangle^+ \propto \frac{Nn}{\ln(n)}, \quad n\to\infty. \tag{19}$$

VI. FINAL RESULTS AND CROSSOVERS BETWEEN DIFFERENT REGIMES

As shown in Sec. II, for $N\to\infty$ the leading term in $\langle S_N(n)\rangle$ is $\langle S_N(n)\rangle^+$, since it is not bounded in contrast to the bounded term $\langle S_N(n)\rangle^-$. Thus the results for $\langle S_N(n)\rangle$ can be approximated by $\langle S_N(n)\rangle^+$ derived in Eqs. (14), (16), and (18) for different values of α. The results for $\alpha<2$ are shown in Fig. 2 to be in good agreement with the Monte Carlo (MC) simulation results.

For the limit $n\to\infty$ we apply an alternative approach. For any value of α the number of visited sites is less than Nn, therefore for $\alpha<2$ one has the estimations

$$Nn \geqslant \langle S_N(n)\rangle \geqslant \langle S_N(n)\rangle^+ \propto Nn. \tag{20}$$

Thus $\langle S_N(n)\rangle \propto Nn$ in the limit $n\to\infty$.

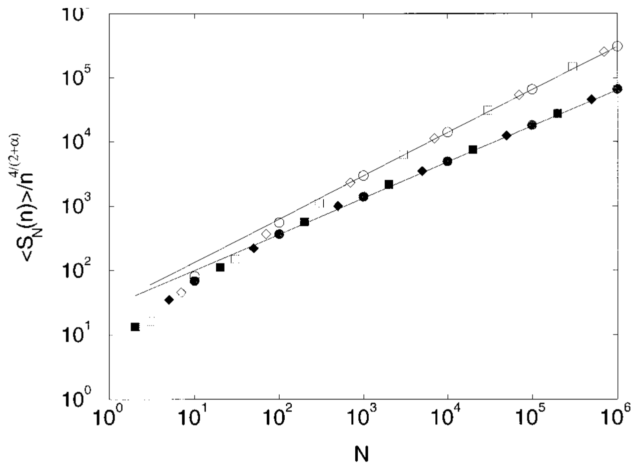

FIG. 2. Results obtained from MC simulations for the N dependence for $\alpha=1.0$ (empty symbols) and $\alpha=1.5$ (filled symbols). Results are scaled by multiplying $\langle S_N(n)\rangle$ by $n^{-4/(2+\alpha)}$. The fitted slopes of the lines are 0.67 and 0.56, which are to be compared to the theoretical values of $2/(2+\alpha)=2/3$ and 0.57, respectively, see Eq. (16). The plotted data correspond to $n=10$ (○), 25 (◇), and 50 (□).

By equating the two expressions Eq. (20) and Eq. (16) for $\langle S_N(n)\rangle$, $\alpha<2$, we can obtain the number of steps $n_\times$ and the number of particles $N_\times$ for the crossover between the two different asymptotic regimes $n\to\infty$ and $N\to\infty$,

$$Nn \propto N^{2/(2+\alpha)}n^{4/(2+\alpha)},$$

$$N_\times \propto n_\times^{(2-\alpha)/\alpha}. \tag{21}$$

For $N=N_\times$ and $n<n_\times$ we expect Eq. (16) to be valid while for $n>n_\times$ we expect $\langle S_N(n)\rangle \propto Nn$. In Fig. 3 the n dependence before the crossover to the Nn regime is shown. The fitted slope of the plots is close to that predicted by Eq.

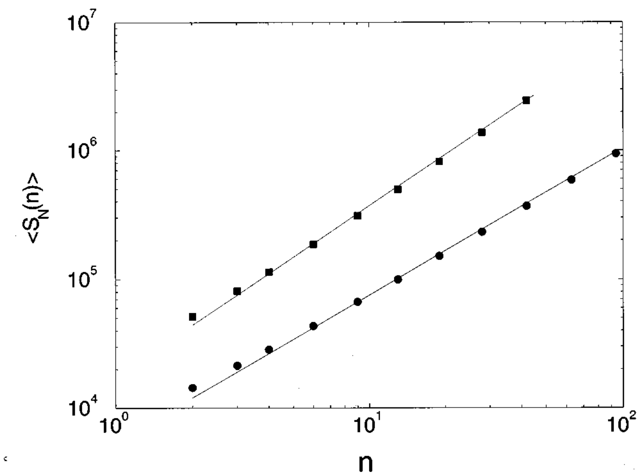

FIG. 3. The n dependence of $\langle S_N(n)\rangle$ with $N=10^6$ plotted for the cases $\alpha=1.0$ (□) and $\alpha=1.5$ (●). The slopes of the fitted lines are 1.32 and 1.14, which are to be compared with the prediction of $4/(2+\alpha)=1.33$ and 1.14, respectively, see Eq. (16).

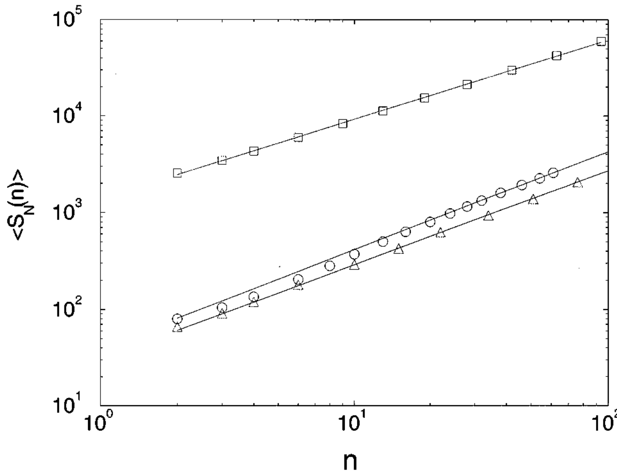

FIG. 4. The n dependence of the function $\langle S_N(n)\rangle$ near the crossover between the regimes $N^{2/(2+\alpha)}n^{2/\alpha}$ and $n\ln(N/\ln n)$, see Table I for the case $\alpha>2$. In the case $\alpha=2.5$ (□) we see the behavior before the crossover, $\langle S_N(n)\rangle \propto n^{2/\alpha}$: the fitted line slope is $0.82\approx 2/\alpha=0.8$. The plot for the case $\alpha=7.0$ (○) shows partially (small n) the crossover regime and the slope for the large n is about 1.0. In the plot for $\alpha=8.0$ (△) the crossover regime does not appear and $\langle S_N(n)\rangle \propto n$: the fitted slope is about 0.97.

(16). From Eq. (21) it is also seen that for $\alpha\to 2^-$ the dependence of type $\langle S_N(n)\rangle \propto Nn$ disappears.

For $\alpha>2$ one can approximate $p_n(\mathbf{r})$ in the circle $|\mathbf{r}|\leqslant n^{1/2}$ by the Gaussian distribution, therefore the derivation of Larralde *et al.* [39] can be employed, yielding

$$\langle S_N(n)\rangle^- \propto \begin{cases} n\ln(N/\ln n), & n<\exp(\gamma N) \quad (22)\\ Nn/\ln n, & n>\exp(\gamma N). \quad (23)\end{cases}$$

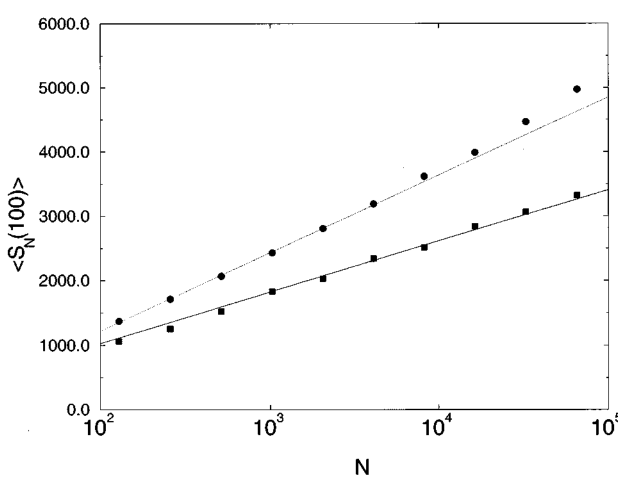

FIG. 5. The N dependence of the function $\langle S_N(n)\rangle$ for $n=100$ is plotted in semilogarithmic scale. In the plot for the case $\alpha=4.0$ (●) one can see at large N the beginning of the crossover between the regimes of $n\ln(N/\ln n)$ and $N^{2/(2+\alpha)}n^{2/\alpha}$, see Table I. The plot for $\alpha=7.0$ (□) shows the behavior before the crossover. For this case $\langle S_N(n)\rangle$ is proportional to $\ln(N)$, as predicted.

TABLE I. Summary of the results for $\langle S_N(n)\rangle$, $d=2$, obtained in the paper.

	$\alpha<2$	$\alpha=2$	$\alpha>2$
$N\to\infty$	$(Nn^2)^{2/(2+\alpha)}$	$N^{1/2}n/\sqrt{\ln(n)}$	$N^{2/(2+\alpha)}n^{2/\alpha}$
$n\to\infty$	Nn	$Nn/\ln[\ln(n)]$	$n\ln(N/\ln n)$, $n<\exp(\gamma N)$ $Nn/\ln n$, $n>\exp(\gamma N)$

Comparing these results with the result of Eq. (15) one can see that the leading term in the sum for $\langle S_N(n)\rangle$ comes from $\langle S_N(n)\rangle^-$.

In the limit $n\to\infty$ the main regime is that of Eq. (23), since Eq. (22) is valid only for bounded n. However, in the following we show that the regime of Eq. (22) is also important.

By comparing Eqs. (22) and (23) with Eq. (14) we can obtain some estimation for the crossover time $n_\times$ and crossover number of particles $N_\times$ between those regimes. In particular, by equating Eqs. (22) and (14) one gets

$$n_\times^{(\alpha-2)/\alpha}=\frac{N_\times^{2/(\alpha+2)}}{\ln(N_\times/\ln n_\times)}$$

and condition $n<\exp(\gamma N)$ implies $N_\times/\ln n_\times>\gamma^{-1}$, from which follows

$$n_\times<[(-\ln\gamma)^{-1}N_\times^{2/(\alpha+2)}]^{\alpha/(\alpha-2)}.$$

Thus for $N_\times$ such that

$$[(-\ln\gamma)^{-1}N_\times^{2/(\alpha+2)}]^{\alpha/(\alpha-2)}<\exp(\gamma N_\times)$$

the regime of Eq. (22) takes place for the values of n: $n_\times<n<\exp(\gamma N_\times)$.

Indeed, MC simulations support the existence of the regime Eq. (22). The crossover between the regimes of Eqs. (22) and (23) and the regime of Eq. (14) is shown in Figs. 4 and 5. In Fig. 4 we present data for the n dependence for N fixed, and in Fig. 5 the N dependence for n fixed. Note that Fig. 5 is in good agreement with Eq. (23).

The case $\alpha=2$ is the boundary case, i.e., the distribution $p(\mathbf{r})\propto|\mathbf{r}|^{-2-\alpha}$ belongs neither to the domain of attraction of the stable laws with infinite variance nor to the domain of the attraction of Gaussian stable law. Thus for $|\mathbf{r}|\leqslant n^{1/2}$ the limit distribution (if one exists) cannot be described by stable laws. Nevertheless, we have the lower bound in Eq. (19) and the natural upper bound Nn. MC simulations suggest that the behavior in the limit $n\to\infty$ is proportional to $Nn/\ln[\ln(n)]$, which obeys the discussed bounds and coincides with the results for $N=1$ [37].

TABLE II. Results predicted for $\langle S_N(n)\rangle$ for $d>2$.

	$\alpha\leqslant 2$	$2<\alpha<d$	$\alpha=d$	$\alpha>d$
$N\to\infty$	$(Nn^2)^{d/(d+\alpha)}$		$N^{1/2}n[\ln(n)]^{-1/2}$	$N^{d/(d+\alpha)}n^{d/\alpha}$
$n\to\infty$	Nn		$[n\ln(Nn^{1-d/2})]^{d/2}$, $n<N^{2/(d-2)}$ Nn, $n>N^{2/(d-2)}$	

VII. SUMMARY AND EXTENSION TO HIGHER DIMENSIONS

The results obtained in this paper are summarized in Table I.

Using the technique presented in this paper and in [40] one can extend the analysis to higher dimensions, $d \geqslant 3$. Namely, the integral (12) converges for $\alpha < d$ and therefore we consider three regimes: $\alpha < d$, $\alpha = d$, and $\alpha > d$, which we treat in complete analogy to the analysis of Sec. III, IV, and V correspondingly. This gives us the results for the $N \to \infty$ limit.

The other limit, $n \to \infty$, has two different regimes: for $\alpha < 2$ the appropriate approximation for $p_n(\mathbf{r})$ is infinite variance stable laws, while for $\alpha > 2$ one should use Gaussian approximation for $|\mathbf{r}| < n^{1/2}$. Thus the formalism for $\alpha < 2$ is analogous to the one from Sec. IV and for $\alpha > 2$ we use results of [39]. Summarizing all the above we get Table II.

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